

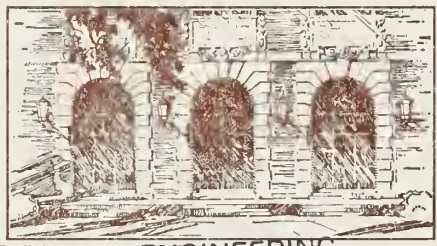


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ON THE IDENTIFICATION OF MULTI-OUTPUT  
LINEAR TIME-INVARIANT AND PERIODIC  
DYNAMIC SYSTEMS

By

Ahmed H. Sameh  
and  
Walter L. Heimerdinger

November 1, 1972





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ABSTRACT

In this paper we describe an efficient computational algorithm for estimating the coefficients of the characteristic polynomial of a linear time-invariant multi-output dynamic system, using only output observations, for qualitative analysis of the transition matrix or for evaluating its eigenvalues. We also give some computational results of the identification of those systems using the Ho-Kalman approach. Furthermore, an identification scheme for high-frequency periodic systems of unknown periods is described in detail.



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## I. INTRODUCTION

Recently much effort has been devoted to the study of mathematical model building for various physical processes. A great deal of data in economics, engineering, biology, and other fields are generated by processes that can be described by multi-output linear time-invariant systems of differential equations  $\dot{x}(t) = Ax(t)$ ,  $z(t) = Hx(t)$ , (where  $A$  and  $H$  do not have necessarily the same number of rows), or periodic systems  $\dot{x}(t) = A(t)x(t)$ ,  $z(t) = H(t)x(t)$ ;  $A(t+p) = A(t)$ ,  $H(t+p) = H(t)$  of high frequency where the period  $p$  is not known but the data at hand contain a large number of periods. In many instances the data observed are noise corrupted versions of the state vectors of the system and it is impractical or impossible to impart input signals to these systems for the sake of control or experimentation so as to aid the identification process, i.e. a passive identification procedure is required.

In this paper we show that the identification of such periodic systems can be reduced to the identification of several time-invariant systems. Consequently the central problem reduces to that of identifying these time-invariant systems using output observations only.

Several methods have been developed for the identification of linear time-invariant systems. They can be classified as follows:

- (i) Linear least squares methods [1 - 3]
- (ii) Instrumental variable methods [4 - 7]
- (iii) Stochastic approximation methods [8 - 13]
- (iv) Maximum likelihood methods [14 - 18]
- (v) Correlation methods [19, 20]

Of the above five methods correlation techniques seem to be the most suitable for identifying linear time-invariant systems using only output observations. Using such a technique we develop an efficient computational algorithm for computing a canonical form of the

transition matrix for qualitative analysis or for obtaining the eigenvalues of the transition matrix of time-invariant systems or the invariants of the monodromy group of the periodic system. We also give some computational results of the identification algorithm of Ho and Kalman [21].

## II. TIME-INVARIANT SYSTEMS

Consider the multi-output linear dynamic system

$$\begin{aligned} x_{k+1} &= \Phi x_k + w_k & k \geq 0 \\ z_k &= Hx_k + v_k & k \geq 1 \end{aligned} \quad (1)$$

where  $x_k$  is an  $n$ -dimensional vector which represents the state of the system at time  $k$ ;  $\Phi$  is an  $n \times n$  time-invariant transition matrix with spectral radius less than unity;  $z_k$  is an  $m$ -dimensional observation vector ( $1 \leq m \leq n$ ),  $H$  is an  $m \times n$  observation matrix of rank  $m$ ;

$v_k$  and  $w_k$  are independent noise sequences of zero mean, uncorrelated, and normally distributed random vectors with finite covariance matrices  $V$  (positive semi-definite) and  $W$  (positive definite), respectively. The initial state vector  $x_0$  is also assumed to be independent of both noise vectors, normally distributed with zero mean, and have a finite positive definite covariance matrix  $P_0$ .

Furthermore, the deterministic system  $x_{k+1} = \Phi x_k$ ,  $z_k = Hx_k$  is assumed to be completely observable, and identifiable [22]; i.e., the  $n \times mn$  matrix

$$\theta^t = \begin{bmatrix} H^t & (H\Phi)^t & \dots & (H\Phi^{n-1})^t \end{bmatrix} \quad (2)$$

and the  $n \times n$  matrix

$$B = \begin{bmatrix} x_0 & \Phi x_0 & \dots & \Phi^{n-1} x_0 \end{bmatrix} \quad (3)$$

are both of rank  $n$ . As we will see later one important implication of the above condition is that the matrix  $\Phi$  must be nonderogatory; i.e., similar to the companion matrix of its characteristic polynomial, [23].

A. Estimation of a canonical form of the transition matrix  $\Phi$ :

From (1) the observation vectors  $z_k$  can be expressed as,

$$z_k = H\Phi^k x_0 + H \sum_{i=0}^{k-1} \Phi^i w_{k-1-i} + v_k \quad (4)$$

Let the  $mn$  - vector  $r_k$  be given by,

$$r_k^t = \begin{bmatrix} z_{k-n+1}^t & : & z_{k-n+2}^t & : & \cdots & : & z_k^t \end{bmatrix} \quad (5)$$

hence, from (4) and (5) we get

$$r_k = \Theta \Phi^{k-n+1} x_0 + \tilde{H} \tilde{w}_{k-1} + \tilde{v}_k \quad (6)$$

where  $\Theta$  is as defined by (2),

$$\tilde{v}_k^t = \begin{bmatrix} v_{k-n+1}^t & : & v_{k-n+2}^t & : & \cdots & : & v_k^t \end{bmatrix},$$

$$\tilde{w}_k^t = \begin{bmatrix} (\sum_{i=0}^{k-n} \Phi^i w_{k-n-i})^t & : & w_{k-n+1}^t & : & \cdots & : & w_{k-1}^t \end{bmatrix},$$

and  $\tilde{H}$  is an  $mn \times n^2$  matrix given by,

$$\tilde{H} = \begin{bmatrix} H & & & & \\ H\Phi & H & & & \\ H\Phi^2 & H\Phi & H & & \\ \vdots & \vdots & \vdots & \ddots & \\ H\Phi^{n-1} & H\Phi^{n-2} & H\Phi^{n-3} & \dots & H \end{bmatrix} \quad (7)$$

Constructing the  $mn \times n$  matrices,

$$R_k = \begin{bmatrix} r_{k-n+1} & : & r_{k-n+2} & : & \cdots & : & r_k \end{bmatrix}$$

$$R_{k+1} = \begin{bmatrix} r_{k-n+2} & : & r_{k-n+3} & : & \cdots & : & r_{k+1} \end{bmatrix}$$

hence from (6) we obtain,

$$R_k = \theta \phi^{k-2n+2} B + \tilde{H}\sigma_{k-1} + \xi_k \quad (8)$$

$$R_{k+1} = \theta \phi^{k-2n+3} B + \tilde{H}\sigma_k + \xi_{k+1}$$

where the  $m \times n$  and  $n^2 \times n$  matrices  $\xi_k$  and  $\sigma_k$  are given by,

$$\xi_k = \begin{bmatrix} \tilde{v}_{k-n+1} & \tilde{v}_{k-n+2} & \cdots & \tilde{v}_k \end{bmatrix}, \quad (9)$$

$$\sigma_k = \begin{bmatrix} \tilde{w}_{k-n+1} & \tilde{w}_{k-n+2} & \cdots & \tilde{w}_k \end{bmatrix}$$

Since the system is identifiable  $B^{-1}$  exists, denoting the matrix  $B^{-1}\phi B$  by  $\Psi$ ; then from (8) and the fact that  $\theta\phi^{j+1}B = (\theta\phi^j B)\Psi$ , it can be easily shown that

$$(R_{k+1} - R_k \Psi) = \tilde{H}(\sigma_{k+1} - \sigma_k \Psi) + (\xi_{k+1} - \xi_k \Psi) \quad (10)$$

Let the  $n \times n$  matrix  $B^{-1}$  be given by  $\begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}^t$ , hence from the fact that  $B^{-1}B = I$  we obtain  $b_i^t(\phi^j x_0) = \delta_{ij}$  for all  $i, j$ .

Consequently the matrix  $\Psi = B^{-1}\phi B$  is of the form,

$$\Psi = \begin{bmatrix} 0 & \vdots & \\ \cdots & \vdots & \\ I & \vdots & \\ (n-1 \times n-1) & \vdots & \end{bmatrix} \psi \quad (11)$$

Thus the problem of estimating the  $n^2$  elements of the transition matrix of an equivalent dynamic system has been reduced to that of only estimating the  $n$ -elements  $\psi_i$  of the vector  $\psi$ . Now from (6), (10), and (11) we get

$$z_{k+n+1} = \sum_{i=1}^n z_{k+i} \psi_i + \omega_{k+n+1} \quad k \geq 1 \quad (12)$$

where,

$$\omega_{k+n+1} = \left[ v_{k+n+1} - \sum_{i=0}^n \psi_i v_{k+i} \right] + \left[ \sum_{i=0}^{k+n} H\phi^i w_{k+n-i} - \sum_{j=1}^n \psi_j \sum_{i=0}^{k+j-1} H\phi^i w_{k+j-i-1} \right] \quad (13)$$

However, since  $\phi$  satisfies its characteristic polynomial, i.e.

$$(-1)^n \left[ \phi^n - \psi_n \phi^{n-1} - \psi_{n-1} \phi^{n-2} - \dots - \psi_1 I \right] = 0 \quad (14)$$

(13) is reduced to

$$\omega_{k+n+1} = \left[ v_{k+n+1} - \sum_{i=1}^n \psi_i v_{k+i} \right] + \left[ \mu_n - \sum_{i=1}^n \psi_i \mu_{i-1} \right] \quad (15)$$

where,

$$\mu_j = \sum_{i=0}^j H\phi^i w_{k+j-i} \quad j = 0, 1, \dots, n \quad (16)$$

Let  $\gamma_k$  and  $\Gamma_k$  be an  $n$ -vector and  $n \times n$  matrix defined by,

$$\gamma_k^t = \left[ z_k^t z_{k+n+1}, z_k^t z_{k+n+2}, \dots, z_k^t z_{k+2n} \right] \quad (17)$$

$$\Gamma_k = \begin{bmatrix} z_k^t z_{k+1} & z_k^t z_{k+2} & \dots & z_k^t z_{k+n} \\ z_k^t z_{k+2} & z_k^t z_{k+3} & \dots & z_k^t z_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_k^t z_{k+n} & z_k^t z_{k+n+1} & \dots & z_k^t z_{k+2n-1} \end{bmatrix} \quad (18)$$

Therefore, from (12) we obtain

$$\gamma_k = \Gamma_k \psi + \eta_k \quad (19)$$

where

$$\eta_k^t = \left[ z_k^t \omega_{k+n+1}, z_k^t \omega_{k+n+2}, \dots, z_k^t \omega_{k+2n} \right] \quad (20)$$



Consider now the expected value of the inner product  $z_k^t \omega_{k+j}$  for  $j \geq n+1$ . From the assumption that  $x_0; v_1, v_2, \dots$ ; and  $w_0, w_1, \dots$  are independent random variables, it can easily be shown that  $E [z_k^t \omega_{k+j}] = 0$  for all  $j \geq n+1$ , i.e.  $E [\eta_k] = 0$ , and

$$E [\gamma_k] = \{E [\Gamma_k]\} \cdot \psi \quad (21)$$

Denoting  $E [z_k^t z_{k+j}]$  by  $c_j$ , then from (21)

$$c_{n+j} = \sum_{k=1}^n c_{j+k} \psi_k \quad j = 1, 2, \dots, n \quad (22)$$

hence the Hankel matrix  $E [\Gamma_k]$  is of rank  $n$  [24], and  $\psi$  is obtained by solving the system of linear equations (21). Motivated by (21), we will estimate  $\psi$  by obtaining the least-squares solution of

$$\left[ \frac{1}{\ell} \sum_{k=1}^{\ell} \gamma_k \right] = \left[ \frac{1}{\ell} \sum_{k=1}^{\ell} \Gamma_k \right] \psi$$

$$\text{i.e.} \quad \hat{\psi}_\ell = \left[ \frac{1}{\ell} \sum_{k=1}^{\ell} \Gamma_k \right]^{\#} \left[ \frac{1}{\ell} \sum_{k=1}^{\ell} \gamma_k \right] \quad (23)$$

where  $A^{\#}$  is the pseudo-inverse of the matrix  $A$ , [25]. Now we introduce the following theorem,

If  $\Phi$  is a matrix of spectral radius  $\rho(\Phi) < 1$ , and if  $W$  is a positive definite matrix, then  $\hat{\psi}_\ell \rightarrow \psi$  a.s. (with probability one) as  $\ell \rightarrow \infty$ .

### Proof

The proof of this theorem is similar to that of Theorem 2.1 of [19].

To prove the above theorem we have to show that

$$\frac{1}{\ell} \sum_{k=1}^{\ell} z_k^t \omega_{k+j} \rightarrow 0 \quad \text{a.s. for } j \geq n+1 \quad (24)$$

and,

$$\frac{1}{\ell} \sum_{k=1}^{\ell} \Gamma_k \rightarrow \hat{\Gamma}_{\ell} \quad \text{a.s.} \quad (25)$$

where  $\hat{\Gamma}_{\ell}$  is of rank  $n$ .

First we establish a bound on the covariance matrix of  $z_k$ .

From (4),

$$E[z_k] = H\Phi^k E[x_0] \quad (26)$$

$$E[z_k z_k^t] = H P_k H^t + V \quad (27)$$

where,

$$P_k = \Phi^k P_0 (\Phi^k)^t + \sum_{i=0}^{k-1} \Phi^i W (\Phi^i)^t$$

Since  $\rho(\Phi) < 1$ , then as  $k \rightarrow \infty$

$$E[z_k z_k^t] \rightarrow H P H^t + V < \infty \quad (28)$$

in which,

$$P = \sum_{i=0}^{\infty} \Phi^i W (\Phi^i)^t \quad (29)$$

From (15) we have

$$\begin{aligned} \sum_{k=1}^{\ell} z_k^t \omega_{k+n+1} &= \sum_{k=1}^{\ell} z_k^t (v_{k+n+1} + \mu_n) - \\ &\quad \sum_{i=1}^n \psi_i \sum_{k=1}^{\ell} z_k^t (v_{k+i} + \mu_{i-1}) \end{aligned} \quad (30)$$

Let us define,

$$\begin{aligned} S_{\ell,1} &= \sum_{k=1}^{\ell} k^{-1} z_k^t v_{k+j} \\ S_{\ell,2} &= \sum_{k=1}^{\ell} k^{-1} z_k^t H \Phi^i w_{k+j-1} \end{aligned} \quad \begin{aligned} j &\geq 1 \\ (31) \end{aligned}$$

Also let  $F_\ell$  be the smallest  $\sigma$ -algebra with respect to which  $x_0; v_1, v_2, \dots, v_{\ell+j}; w_0, w_1, \dots, w_{\ell+j-1}$  ( $j \geq 1$ ) are measurable. Since,

$$\lim_{\ell \rightarrow \infty} E[|S_{\ell,1}|] = \lim_{\ell \rightarrow \infty} \sum_{k=1}^{\ell} k^{-1} E[|z_k^t v_{k+j}|] = 0 < \infty \quad (32)$$

and,

$$E[S_{\ell+1,1}|F_\ell] - S_{\ell,1} = \frac{1}{\ell+1} E[z_{\ell+1}^t v_{\ell+j+1}|F_\ell] = 0 \quad (33)$$

where we have used the fact that  $z_{\ell+1}$  is measurable w.r.t.  $F_\ell$  and that  $v_{\ell+j+1}$ ,  $j \geq 1$ , is independent of  $F_\ell$ . Therefore,  $\{S_{\ell,1}; F_\ell\}$  is a martingale [26]. Similarly, it can be shown that  $\{S_{\ell,2}; F_\ell\}$  is also a martingale. Moreover,

$$E[S_{\ell,1}^2] = E\left[\sum_{k=1}^{\ell} k^{-1} z_k^t v_{k+j}\right]^2 = \sum_{k=1}^{\ell} k^{-2} \text{tr}[V(HP_k H^t + V)]$$

and,

$$\begin{aligned} E[S_{\ell,2}^2] &= E\left[\sum_{k=1}^{\ell} k^{-1} z_k^t H\Phi^i w_{k+j-1}\right]^2 \\ &= \sum_{k=1}^{\ell} k^{-2} \text{tr}[(HP_k H^t + V) H\Phi^i W(\Phi^i)^t] \end{aligned}$$

are both bounded by virtue of (28).

Thus by the martingale convergence theorem [26]  $\lim_{\ell \rightarrow \infty} S_{\ell,i}$  exists and is finite a.s., ( $i = 1, 2$ ). Also from the Kronecker Lemma [27], we obtain

$$\frac{1}{\ell} \sum_{k=1}^{\ell} z_k^t v_{k+j} \rightarrow 0 \quad \text{a.s.} \quad (34)$$

$$\frac{1}{\ell} \sum_{k=1}^{\ell} z_k^t H\Phi^i w_{k+j-1} \rightarrow 0 \quad \text{a.s.}$$

Hence from (16) and (30) it is clear that (24) is satisfied.

Let  $w_{-1}, w_{-2}, \dots$  be a sequence of independent random vectors which have the same distribution as the  $w_k$ 's and are mutually independent of  $x_0, w_0, w_1, \dots; v_1, v_2, \dots$ ; such a sequence may always be found by enlarging the probability space [26]. Now (4) could be written as,

$$\begin{aligned} z_k &= \left[ v_k + \sum_{i=0}^{\infty} H\Phi^i w_{k-i-1} \right] - \left[ H\Phi^k \left( \sum_{i=0}^{\infty} \Phi^i w_{-i-1} - x_0 \right) \right] \\ &= u_k - q_k \end{aligned} \quad (35)$$

Since  $\rho(\Phi) < 1$ , then by the Three Series Theorem [26] it can be proved that  $u_k$  and  $q_k$  are well defined random variable. Furthermore  $\lim_{k \rightarrow \infty} \|q_k\| = 0$  a.s. and in the mean square, and  $\sum_{i=0}^{\infty} \|H\Phi^i\|^2 < \infty$ .

Consequently  $u_k$  is a moving average of the  $w_i$ 's and  $v_j$ 's; therefore  $u_k$  and  $u_k^t u_{k+j}^t$  ( $j \geq 0$ ) are metrically transitive strictly stationary processes. From (35) we have,

$$z_k^t z_{k+j}^t = u_k^t u_{k+j}^t + q_k^t q_{k+j}^t - u_k^t q_{k+j}^t - q_k^t u_{k+j}^t \quad j \geq 0$$

Since  $\rho(\Phi) < 1$ ,  $\lim_{k \rightarrow \infty} \|q_k\| = 0$  a.s., and

$$\sup_{k \geq 1} k^{-1} \|u_k\| \leq \sup_{N \geq 1} N^{-1} \sum_{j=1}^N \|u_j\| < \infty \quad \text{a.s.},$$

we see that,

$$\lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{k=1}^{\ell} z_k^t z_{k+j}^t = \lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{k=1}^{\ell} u_k^t u_{k+j}^t \quad \text{a.s.} \quad (36)$$

Furthermore, from (35) and (29) we get

$$\begin{aligned} E[u_k^t u_{k+j}^t] &= \text{tr} \left[ \Phi^j \sum_{i=0}^{\infty} H\Phi^i W(\Phi^i)^t H^t \right] \\ &= \text{tr} [\Phi^j H P H^t] < \infty \end{aligned} \quad (37)$$

Thus from the ergodic theorem, (36), and (37), we have for  $j \geq 0$

$$\lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{k=1}^{\ell} z_k^t z_{k+j} = E[u_k^t u_{k+j}] = E[z_k^t z_{k+j}] < \infty \quad \text{a.s.} \quad (38)$$

As a result,

$$\lim_{\ell \rightarrow \infty} \ell^{-1} \sum_{k=1}^{\ell} \Gamma_k = \hat{\Gamma}_{\ell} = E[\Gamma_k] < \infty \quad \text{a.s.} \quad (39)$$

and from (22) we see that  $\hat{\Gamma}_{\ell}$  is of rank  $n$  a.s. This completes the proof of the theorem.

#### B. The HO-Kalman approach:

If, however, accurate determination of the eigenvalues of  $\Phi$  is rather sensitive to small perturbations of the coefficients of its characteristic polynomial (elements of  $\psi$ ), or if we desire to completely identify an equivalent dynamic system,

$$\begin{aligned} y_{k+1} &= \Phi^* y_k + w_k^* \\ z_k &= H^* y_k + v_k \end{aligned} \quad (40)$$

where  $y_k = S^{-1}x_k$  (in which  $S$  is an  $n \times n$  nonsingular matrix), we use the identification scheme described by [20, 21]. By complete identification we mean obtaining strong consistent estimates of  $\Phi^* = S^{-1}\Phi S$ ,  $H^* = HS$ ,  $V$ , and the state vectors  $y_k$ .

Let the  $m \times m$  matrix  $E[z_{k+j} z_k^t]$  be defined by,

$$C_j = E[z_{k+j} z_k^t] \quad (41)$$

Assuming that the identification scheme has started after the system reached the steady state, then from (4) we obtain

$$C_j = H\Phi^j P H^t + V \delta_{j,0} \quad (42)$$

where  $P$  is an  $n \times n$  positive definite matrix given by (29). Using (41) we now construct the generalized  $mn \times mn$  Hankel matrices

$$G_{2n-1} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ c_2 & c_3 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \dots & c_{2n-1} \end{bmatrix}, \quad G_{2n} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ c_2 & c_3 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{bmatrix} \quad (43)$$

From (42) we see that

$$G_{2n-1} = \Theta \Phi M, \quad G_{2n} = \Theta \Phi^2 M \quad (44)$$

where  $M$  is an  $n \times mn$  matrix given by,

$$M = [PH^t : \Phi PH^t : \dots : \Phi^{n-1} PH^t] \quad (45)$$

which is of rank  $n$  due to the observability assumption. Consequently

$G_{2n-1}$  and  $G_{2n}$  are also of rank  $n$ . Therefore there exists an orthogonal matrix  $R$  and a permutation matrix  $\pi$  such that

$$RG_{2n-1}\pi = \begin{bmatrix} L_1 & \vdots & L_2 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{bmatrix} \quad (46)$$

where  $L_1$  is an  $n \times n$  nonsingular upper triangular matrix. In practice  $R$  is chosen as the product of  $n$  elementary Hermitian matrices, [23]. Define the  $mn \times mn$  matrix  $Q$  by

$$Q = \begin{bmatrix} L_1^{-1} & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{bmatrix}$$

then we have,

$$RG_{2n-1}\pi Q = \begin{bmatrix} I_n & \vdots & 0 \\ \dots & \vdots & \dots \\ 0 & \vdots & 0 \end{bmatrix} \quad (47)$$

From (44) the above equation can be written as,

$$U\Phi S = I_n \quad (48)$$

where  $U$  and  $S$  are nonsingular  $n \times n$  matrices given by,

$$\begin{aligned} U &= E_n R \Theta \\ S &= M \pi Q E_n^t \end{aligned} \quad (49)$$

in which  $E_q = [I_q : 0]$  is a  $q \times mn$  matrix. The matrices  $\Phi^*$ ,  $H^*$ , and  $P^* H^{*t}$  of the system (40) can be determined as follows,

$$\begin{aligned} \Phi^* &= E_n R G_{2n} \pi Q E_n^t \\ &= E_n R \Theta \Phi^2 M \pi Q E_n^t \\ &= U \Phi^2 S \\ &= S^{-1} \Phi S, \end{aligned} \quad (50)$$

$$\begin{aligned} H^* &= (E_m G_{2n-1} \pi Q E_n^t) \Phi^{*-1} \\ &= (E_m \Theta \Phi M \pi Q E_m^t) (S^{-1} \Phi^{-1} S) \\ &= HS, \end{aligned} \quad (51)$$

and,

$$\begin{aligned} P^* H^{*t} &= E_n R G_{2n-1} E_m^t \\ &= E_n R \Theta \Phi M E_m^t \\ &= U \Phi P H^t \\ &= (S^{-1} P S^{-t}) (S^t H^t) \end{aligned} \quad (52)$$

Finally the observation covariance matrix  $V$  can be readily determined by,

$$V = C_0 - H^* P^* H^{*t} \quad (53)$$



Using recursive filtering [28] the above information (50 - 53) is then sufficient to estimate the optimal gain matrix of the Kalman filter which is used to estimate the state vectors  $y_k$ .

Similar to the proof of the previous theorem we can show that an identification scheme can be constructed by (43 - 52) where the  $m \times m$  submatrices  $C_j = E[z_{k+j} z_k^t]$ ,  $j = 1, 2, \dots, 2n$ , in the generalized Hankel matrices (43) are replaced by,

$$\hat{C}_j^{(\ell)} = \ell^{-1} \sum_{k=1}^{\ell} z_{k+j} z_k^t \quad j = 1, 2, \dots, 2n \quad (54)$$

where  $\ell$  is the size of the sample. If desired one could also obtain estimates of the covariance matrices  $P^* = S^{-1} P S^{-t}$  and  $W^* = S^{-1} W S^{-t}$  from the matrix equations  $P^* H^{*t} = K$  and  $P^* = \Phi^* P^* \Phi^{*t} + W^*$  [29], where  $K = E_n R \hat{G}_{2n-1} E_m^t$ .

### III. PERIODIC SYSTEMS

Consider the following periodic system of order  $n$ ,

$$\begin{aligned}\dot{x}(t) &= A(t) x(t) \\ z(t) &= H(t) x(t)\end{aligned}\tag{55}$$

where  $A(t)$  is an  $n \times n$  periodic matrix of period  $p$ , and  $H(t)$  is an  $m \times n$  matrix which is either time-invariant or periodic with the same period. We also assume that the system (55) is completely observable [30], and asymptotically stable, [31].

Let  $\Phi(t)$  be the fundamental matrix of the system (55), hence

$$\dot{\Phi}(t) = A(t) \Phi(t)\tag{56}$$

and,

$$\Phi(t+p) = \Phi(t) D_{\Phi}\tag{57}$$

where  $D_{\Phi}$  is a unique nonsingular constant matrix called the monodromy matrix of  $\Phi(t)$ , [32], and can always be expressed as

$$D_{\Phi} = e^{pL}\tag{58}$$

in which  $L$  is a constant matrix. Since the system (55) is asymptotically stable then the eigenvalues  $\lambda_i$  ( $i = 1, 2, \dots, n$ ) of  $D_{\Phi}$  are of modulus less than unity,  $|\lambda_i| < 1$ . The eigenvalues of  $D_{\Phi}$  are called the invariants of the monodromy group of the periodic system or the multipliers associated with  $A(t)$ . Let us now introduce the Liapunov transformation [24],

$$x(t) = \Phi(t) e^{-tL} y(t)\tag{59}$$

Substituting (59) in (55) we obtain the system,

$$\begin{aligned}\dot{y}(t) &= L y(t) \\ z(t) &= \tilde{H}(t) y(t)\end{aligned}\tag{60}$$

where,

$$\tilde{H}(t) = H(t) \Phi(t) e^{-tL}\tag{61}$$

which can be easily shown to be periodic of period  $p$ . Let  $p = N \Delta t$  where  $\Delta t$  is a fairly small time increment. Assuming that the observations  $z(t)$  are measured starting at time  $t_0$  and assuming that  $p$  is known, then the identification of the periodic system (55) is reduced to the identification of the  $N$  time invariant systems

$$\begin{aligned} y(t_k + p) &= D_\phi y(t_k) & k = 0, 1, \dots, N-1 \\ z(t_k) &= \tilde{H}(t_k) y(t_k) \end{aligned} \quad (62)$$

where  $t_k = t_0 + k \Delta t$ ,  $D_\phi$  is the time-invariant transition matrix for all  $k$  with  $\rho(D_\phi) < 1$ , and for a given  $t_k$   $\tilde{H}(t_k)$  is invariant under a time step of  $p$ ; i.e.,

$$\begin{aligned} y(t_k + (\alpha+1)p) &= D_\phi y(t_k + \alpha p) \\ z(t_k + \alpha p) &= \tilde{H}(t_k) y(t_k + \alpha p) \end{aligned} \quad \alpha = 0, 1, 2, \dots$$

Let,

$$\begin{aligned} B_{n-1}(t_k) &= \begin{bmatrix} y(t_k) & y(t_k + p) & \dots & y(t_k + (n-1)p) \end{bmatrix} \\ B_n(t_k) &= \begin{bmatrix} y(t_k + p) & y(t_k + 2p) & \dots & y(t_k + np) \end{bmatrix} \end{aligned} \quad (63)$$

therefore,

$$B_n(t_k) = D_\phi B_{n-1}(t_k) \quad (64)$$

We then say that the periodic system (55) is identifiable if  $B_{n-1}(t)$  is nonsingular, i.e. for any  $t_k$   $D_\phi$  is uniquely determined by  $D_\phi = B_n(t_k) B_{n-1}^{-1}(t_k)$ . Assuming that this identifiability condition holds then we have,

$$\begin{aligned} B_{n-1}^{-1}(t_k) D_\phi B_{n-1}(t_k) &= B_{n-1}^{-1}(t_k) B_n(t_k) \\ &= \begin{bmatrix} 0 & \vdots \\ \dots & \vdots \\ I & \vdots \\ (n-1 \times n-1) & \vdots \end{bmatrix} \phi \end{aligned} \quad (65)$$

Define the  $mn$ -vector  $\begin{bmatrix} z^t(t_k) : z^t(t_{k+p}) : \dots : z^t(t_{k+(n-1)p}) \end{bmatrix}$  by  $r_{n-1}^t(t_k)$ , then from (62) we have

$$\begin{aligned} r_{n-1}(t_k) &= \begin{bmatrix} \tilde{H}(t_k) \\ \tilde{H}(t_{k+p}) D_\phi \\ \vdots \\ \tilde{H}(t_{k+(n-1)p}) D_\phi^{n-1} \end{bmatrix} y(t_k) \\ &= \tilde{\Theta}(t_k) y(t_k) \end{aligned} \quad (66)$$

Since the periodic system (55) is completely observable we can easily verify that the  $mn \times n$  matrix  $\tilde{\Theta}(t_k)$ , for all  $k$ , is of rank  $n$ .

Assuming the existence of both plant and observation noise in system (55) then we deal with the identification of the  $N$  time-invariant systems (under a time step of  $p$ ),

$$\begin{aligned} y(t_{k+p}) &= D_\phi y(t_k) + w(t_k) \\ z(t_k) &= \tilde{H}(t_k) y(t_k) + v(t_k) \end{aligned} \quad k = 0, 1, \dots, N-1 \quad (67)$$

where the noise vectors  $w(t_k)$  and  $v(t_k)$  have the same statistical properties as in (1). Provided that the period  $p$  is known, then similar to Section II we can either obtain the companion matrix of  $D_\phi$  by only estimating the  $n$  elements of the vector  $\phi$  in (65) and hence the invariants of the monodromy group (eigenvalues of  $D_\phi$ ), or by using the Ho-Kalman approach together with recursive filtering to completely identify the  $N$  systems in (67). Now we have only to describe a method for estimating the period  $p$ . We will see that an algorithm for estimating  $p$  will also give an estimate of  $\phi$ .

Similar to (12) we obtain the following expression,

$$z(t_{k+(n+1)p}) = \sum_{i=1}^n \phi_i z(t_{k+ip}) + \omega(t_{k+(n+1)p}) \quad (68)$$

where.

$$\begin{aligned} E[\omega(t_k + jp)] &= 0 & j &\geq 0 \\ E[z^t(t_k) \omega(t_k + jp)] &= 0 & j &\geq n+1 \end{aligned}$$

Defining the  $n$ -dimensional vector  $\gamma(t_k)$  and the  $n \times n$  matrix  $\Gamma(t_k)$  by,

$$\gamma^t(t_k) = \begin{bmatrix} \tau_{n+1}(t_k), \tau_{n+2}(t_k), \dots, \tau_{2n}(t_k) \end{bmatrix}$$

and

$$\Gamma(t_k) = \begin{bmatrix} \tau_1(t_k) & \tau_2(t_k) & \dots & \tau_n(t_k) \\ \tau_2(t_k) & \tau_3(t_k) & \dots & \tau_{n+1}(t_k) \\ \vdots & \vdots & & \vdots \\ \tau_n(t_k) & \tau_{n+1}(t_k) & \dots & \tau_{2n}(t_k) \end{bmatrix} \quad (69)$$

where,

$$\tau_j(t_k) = z^t(t_k) z(t_k + jp) \quad (70)$$

Then similar to (23) we estimate the  $n$ -dimensional vector  $\phi$  by,

$$\hat{\phi}_\ell = \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} \Gamma(t_k + i\hat{p}) \right]^\# \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} \gamma(t_k + i\hat{p}) \right] \quad (71)$$

Choosing various values of  $\hat{p}$ ; i.e.  $\hat{p} = \alpha \Delta t$  ( $\alpha = 1, 2, 3, \dots$ ) we obtain various estimates  $\hat{\phi}_\ell(\alpha)$ . If we obtain two identical estimates  $\hat{\phi}_\ell(\alpha_1)$  and  $\hat{\phi}_\ell(\alpha_2)$ , ( $\|\hat{\phi}_\ell(\alpha_1) - \hat{\phi}_\ell(\alpha_2)\| \leq \varepsilon$  where  $\varepsilon$  is a small number), and  $\alpha_2 = 2\alpha_1$  then  $p = \alpha_1 \Delta t$ . A parallel computer such as the Illiac IV [33, 34] is an ideal tool for fast estimation of the period and the identification of the  $N$  time-invariant systems (67).

#### IV. NUMERICAL EXAMPLES

##### Example 1

We considered the following fourth-order single output system,

$$x_{k+1} = \Phi x_k + \Gamma w_k$$

$$z_k = H x_k + v_k$$

$\Gamma^t$  is a known 4-dimensional vector given by (0, 1, 0, 1) and where the sequences  $w_k$  and  $v_k$  are independent normally distributed with zero mean and variances 1.0 and 0.25 respectively. The system ( $\Phi, H$ ) is observable where,

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & -0.656 \\ 1 & 0 & 0 & 0.784 \\ 0 & 1 & 0 & 0.180 \\ 0 & 0 & 1 & 1.000 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

and the eigenvalues of  $\Phi$  are  $0.9 \pm 0.1i$ ,  $-0.4 \pm 0.8i$ . Using only the output observations  $z_k$  the following estimates were obtained for a sample size  $\ell = 1000, 5000$

(i)  $\ell = 1000$

$$\hat{\psi}^t = [-0.655, 0.756, -0.200, 1.060],$$

$$\|\tilde{\psi}\|_2 / \|\psi\|_2 = .048, \text{ (where } \tilde{\psi} = \hat{\psi} - \psi \text{ and } \|\psi\|_2 = [\sum_{i=1}^n \psi_i^2]^{\frac{1}{2}} \text{)},$$

The eigenvalues obtained from the estimated companion matrix are:

$$\lambda(\hat{\Psi}): 0.917 \pm 0.092i, -0.387 \pm 0.788i,$$

$$\hat{\Phi}^* = \begin{bmatrix} 0.959 & -0.693 & 0.760 & -0.206 \\ 0.015 & 0.690 & -0.139 & 0.948 \\ 0 & -0.356 & -0.637 & 0.651 \\ 0 & 0 & -0.891 & 0.048 \end{bmatrix},$$

$$\lambda(\hat{\Phi}^*) : 0.917 \pm 0.094i, \quad -0.387 \pm 0.788i,$$

$$\hat{H}^* = [-0.534, -0.773, 1.370, -1.899],$$

$$\hat{V} = .100 ; \quad |V - \hat{V}|/V = 0.60 ,$$

$$(\hat{P}^* \hat{H}^{*t})^t = [-76.0, 0, 0, 0] .$$

$$(ii) \quad \ell = 5000$$

$$\hat{\psi}^t = [-0.663, 0.757, -0.175, 1.029] ,$$

$$\|\tilde{\psi}\|_2 / \|\psi\|_2 = .028 ,$$

$$\lambda(\hat{\Psi}) : 0.913 \pm 0.112i, \quad -0.398 \pm 0.791i ,$$

$$\hat{\Phi}^* = \begin{bmatrix} 0.950 & -0.727 & 0.729 & -0.164 \\ 0.019 & 0.755 & -0.124 & 0.872 \\ 0 & -0.278 & -0.655 & 0.711 \\ 0 & 0 & -0.870 & -0.021 \end{bmatrix} ,$$

$$\lambda(\hat{\Phi}^*) : 0.913 \pm 0.111i, \quad -0.398 \pm 0.791i,$$

$$\hat{H}^* = [-0.539, -0.926, 1.243, -1.836] ,$$

$$\hat{V} = 0.225; \quad |V - \hat{V}|/V = 0.10 ,$$

$$(\hat{P}^* \hat{H}^{*t})^t = [-67.7, 0, 0, 0] .$$

### Example 2

Consider the fourth order multi-output system,

$$x_{k+1} = \Phi x_k + w_k$$

$$z_k = x_k + v_k$$

where the covariance matrices of  $w_k$  and  $v_k$  are given by  $0.1I$  and  $0.5I$  respectively, where  $I$  is the identity matrix, and the transition matrix  $\Phi$  is



$$\Phi = \begin{bmatrix} 1.512 & 0.738 & 1.137 & 0.426 \\ 1.728 & -0.978 & 2.603 & -0.416 \\ -0.840 & -1.160 & -0.840 & -0.660 \\ -3.056 & 2.056 & -3.556 & 1.306 \end{bmatrix}$$

which was obtained by a similarity transformation on that of Example 1.

For a sample size  $\ell = 5000$  we obtain the following estimates,

$$\hat{\psi}^t = [-0.673, 0.827, -0.216, 1.014] ,$$

$$\|\tilde{\psi}\|_2 / \|\psi\|_2 = .042 ,$$

$$\lambda(\hat{\Psi}): \quad 0.902 \pm 0.094i, -0.395 \pm 0.814i ,$$

$$\hat{\Phi}^* = \begin{bmatrix} 1.123 & -0.431 & 0.384 & 0.201 \\ 0.366 & -0.010 & -0.092 & -0.745 \\ 0.299 & -0.709 & 0.126 & -0.554 \\ -0.047 & -0.280 & 1.072 & -0.226 \end{bmatrix} ,$$

$$\lambda(\hat{\Phi}^*) : \quad 0.898 \pm 0.090i, -0.392 \pm 0.813i ,$$

$$\hat{H}^* = \begin{bmatrix} -0.154 & 0.368 & -0.148 & 0.035 \\ -0.059 & -0.516 & 0.782 & -0.341 \\ 0.102 & -0.417 & 0.090 & -0.096 \\ 0.175 & 1.092 & -1.284 & 0.222 \end{bmatrix} ,$$

$$\hat{V} = \begin{bmatrix} 0.531 & 0.011 & 0 & 0 \\ 0.002 & 0.578 & 0 & 0 \\ 0 & 0 & 0.525 & 0.007 \\ 0 & 0 & 0.079 & 0.547 \end{bmatrix} ,$$

$$\|\hat{V} - V\|_F / \|V\|_F = 0.128, \text{ (where } \|\cdot\|_F \text{ is the Frobenius norm, } \|V\|_F^2 = \sum_{i,j} v_{ij}^2 \text{)},$$

$$\|\hat{H}^{*-1} \hat{\Phi} \hat{H}^* - \hat{\Phi}^*\|_F / \|\Phi\|_F = .039 ,$$

$$\hat{P}^* \hat{H}^{*t} = \begin{bmatrix} -8.489 & -57.362 & -14.469 & 153.753 \\ 0 & -22.609 & -9.494 & 63.047 \\ 0 & 0 & -2.700 & 5.471 \\ 0 & 0 & 0 & -7.162 \end{bmatrix}.$$

### Example 3

Same as Example 2 except that  $W = V = 0.5I$ . For  $\ell = 5000$  the estimates are:

$$\hat{\psi}^t = [-0.651, 0.785, -0.145, 0.958],$$

$$\|\tilde{\psi}\|_2 / \|\psi\|_2 = .038 ,$$

$$\lambda(\hat{\Psi}): 0.891 \pm 0.091i, -0.412 \pm 0.801i,$$

$$\hat{\Phi}^* = \begin{bmatrix} 1.115 & -0.390 & 0.374 & 0.249 \\ 0.395 & -.056 & -0.063 & -0.744 \\ 0.319 & -0.717 & 0.117 & -0.503 \\ -0.060 & -0.270 & 1.060 & -0.199 \end{bmatrix},$$

$$\|\hat{\Phi}^* - \hat{\Phi}_{\text{Ex. 2}}^*\|_F = 0.110,$$

$$\lambda(\hat{\Phi}^*) = 0.893 \pm 0.090i, -0.405 \pm 0.800i,$$

$$\hat{H}^* = \begin{bmatrix} -0.161 & 0.371 & -0.158 & .013 \\ -0.060 & -0.510 & 0.796 & -0.318 \\ 0.113 & -0.432 & 0.106 & -0.091 \\ 0.172 & 1.083 & -1.311 & 0.183 \end{bmatrix},$$

$$\|\hat{H}^{*-1} \hat{\Phi} \hat{H}^* - \hat{\Phi}^*\|_F / \|\Phi\|_F = .004 ,$$

$$\|\hat{H}^* - \hat{H}_{\text{Ex. 2}}^*\|_F = 0.059 ,$$

$$\hat{v} = \begin{bmatrix} 0.574 & 0.036 & 0 & 0 \\ 0 & 0.487 & 0.051 & 0 \\ 0 & 0 & 0.540 & 0 \\ 0.046 & 0.048 & 0 & 0.546 \end{bmatrix},$$

$$\|\hat{v} - v\|_F / \|v\|_F = 0.132,$$

$$\hat{P}^{*t} H = \begin{bmatrix} -39.309 & -272.211 & -69.059 & 727.879 \\ 0 & -112.063 & -48.036 & 311.346 \\ 0 & 0 & -13.973 & 28.173 \\ 0 & 0 & 0 & -37.943 \end{bmatrix}.$$

#### Example 4

Same as Example 3 except that

$$z_k = Hx_k + v_k$$

where,

$$H = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

For  $\ell = 5500$  the estimates are:

$$\hat{\psi}^t = [-0.687, 0.819, -0.158, 0.971],$$

$$\|\tilde{\psi}\|_2 / \|\psi\|_2 = .041,$$

$$\lambda(\hat{\psi}): 0.901 \pm 0.104i, -0.416 \pm 0.813i,$$

$$\hat{\phi}^* = \begin{bmatrix} 0.947 & -0.660 & 0.484 & 0.068 \\ 0.017 & 0.808 & -0.084 & -0.354 \\ -0.004 & -0.349 & -0.403 & -0.640 \\ 0 & 0.111 & 0.914 & -0.366 \end{bmatrix},$$

$$\lambda(\hat{\phi}^*): 0.902 \pm 0.104i, -0.409 \pm 0.809i,$$

$$\hat{H}^* = \begin{bmatrix} -0.531 & -1.186 & 0.133 & 0.746 \\ 0.109 & -0.228 & -0.711 & -0.257 \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} 0.584 & 0 \\ 0 & 0.566 \end{bmatrix},$$

$$\|\hat{V} - V\|_F / \|V\|_F = 0.151,$$

$$\hat{P}^* \hat{H}^* \hat{t} = \begin{bmatrix} -341.627 & 80.054 \\ 0 & -4.714 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

### Example 5

In order to illustrate the method for finding the period of a periodic system which is both observable and identifiable, we consider for the sake of simplicity the deterministic system

$$\frac{dx}{dt} = A(t) x(t)$$

where,

$$A(t) = [\text{diag}(-0.95, -0.90, -0.85)] \sin \omega t$$

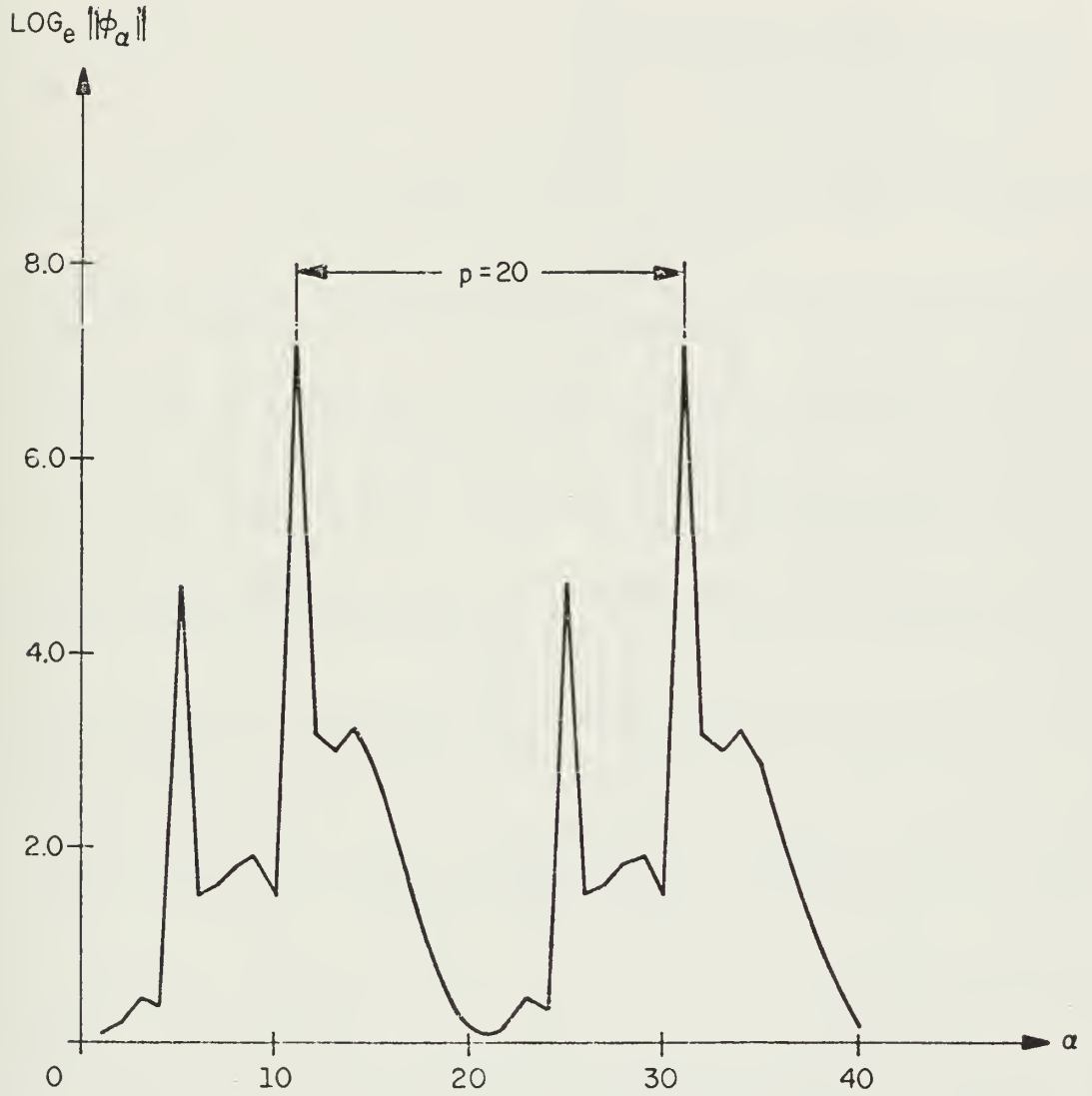
in which  $\omega = \pi/10$ . Starting with the initial conditions  $x^t(0) = (1, 1, 1)$  we generated 50 observation vectors  $x(t)$  with  $\Delta t = 1$  sec.

Applying algorithm (71), with  $t_k = 0$ , we plotted  $\log_e \|\phi_\alpha\|$  versus  $\alpha$  as shown in the figure below. The period is clearly 20 sec. and  $\|\phi\|_2 = 1.175$  where,

$$\phi^t = (.088, -0.464, 1.076) \quad (72)$$

The invariants of the monodromy group of this periodic system are the eigenvalues of (65) with  $\phi$  as in (72),

$$\lambda_{1,2} = 0.288 \pm 0.305i, \lambda_3 = 0.500.$$



## V. CONCLUSION

The assumptions of observability and identifiability of linear time-invariant multi-output dynamic systems have been effectively used to estimate a canonical form of the transition matrix for qualitative analysis or for evaluating the invariants of the system using only output data. The computational algorithm is time-saving since we estimate  $n$  elements only rather than the  $n^2$  elements of the transition matrix either of the system itself or of an equivalent one. The results show that these coefficients can be determined fairly accurately and the eigenvalues of the companion matrix agree reasonably well with those of the actual transition matrix. Again using only output observations the Ho-Kalman approach was implemented to obtain consistent estimates of  $\Phi^*$ , and  $H^*$  of an equivalent dynamic system together with the observation noise covariance matrix  $V$ . Adaptive filtering could then be used for direct estimation of the optimal gain of the Kalman filter which we may use to estimate the state vectors of the equivalent dynamic system.

Finally we showed that under certain assumptions the problem of identification of a high-frequency periodic system of unknown period could be reduced to the identification of several time-invariant dynamic systems and estimation of the period.

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| 13. ABSTRACT<br><br>In this paper we describe an efficient computational algorithm for estimating the coefficients of the characteristic polynomial of a linear time-invariant multi-output dynamic system, using only output observations, for qualitative analysis of the transition matrix or for evaluating its eigenvalues. We also give some computational results of the identification of those systems using the Ho-Kalman approach. Furthermore, an identification scheme for high-frequency periodic systems of unknown periods is described in detail. |  |                                                                                                               |                       |

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